AN ANALYTIC SOLUTION TO THE NEUGEBAUER EQUATIONS

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Abstract

For over 50 years the Neugebauer equations have provided a model for color reproduction in halftone screened images. But because the equations are nonlinear, they have been solved either by analogue computation or numerical digital techniques. Analytic approximations can be made only through severe simplifying assumptions. The equations originally described the printing with three inks (CMY), but can easily be extended to the case of a fourth black ink (CMYK). This appears to complicate the situation, making the equations even more nonlinear, increasing the number of equations, and making the problem nondeterministic. However, in one case (the case of full undercolor removal) the presence of a very good black ink can actually simplify the problem to the point where it admits an analytic solution.

This paper provides an analytic method for determining dot areas which satisfy the Neugebauer equations for the case of full undercolor removal. The paper also presents a brief review of previous methods for the solution of the equations, and comments on the inclusion of Yule-Neilsen correction.

Introduction

A classic problem in electronic printing is how to determine the amounts of the various inks required to produce a desired color. The recent developments of inexpensive, networked color workstations and printers have focused renewed attention on device independent color specification, device calibration and on the production of a desired color. One approach is to model the behavior of the printer so that a prediction of the color that will result from a given specification of ink amounts can be made. The Neugebauer equations provide such a model for the case of rotated halftone screens. but numerical methods or approximations are needed to solve them. This paper describes an analytic method for solving the Neugebauer equations for the special case of full undercolor removal and

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perfect black. It also briefly considers the Yule-Neilsen correction to the Neugebauer equations.

Background

In full color printing the various colors are produced by combinations of three primary inks, cyan, magenta and yellow. The cyan absorbs red, the magenta absorbs green and the yellow absorbs blue. If the inks were ideal, then it would not matter whether or not they overlapped; a cyan dot next to a magenta dot would yield the same color as a cyan dot on top of a magenta dot. However real inks are not ideal and the colors where inks overlap can be quite different from the colors of the same ink amounts in a nonoverlapping configuration. The color printed must be thought of as arising from eight colors (not just three) corresponding to the eight possible ink combinations. They are white (no ink) cyan, magenta, and yellow (single inks), red, green, blue (pairs of inks), and black (all three inks). The sum of the color coordinates of these eight colors weighted by the amounts of these colors should yield the coordinates of the color actually seen. However, due to errors in registration when printing the halftone dot patterns for each of the color separations one usually does not know the exact amounts of each overlap configuration.

An approach to making the colors more predictable would be to randomly distribute the dot patterns. If the dot positions are sufficiently distributed, the average amount of a primary color, secondary color, white or black is given by the probability of finding the various inks, and those probabilities are directly related to the ink areas. If $\bar{W}$, $\bar{C}$, $\bar{M}$, $\bar{J}$, $\bar{R}$, $\bar{G}$, $\bar{B}$, $\bar{K}$ are the visible white, cyan, magenta, yellow, red, green, blue, and black fractional dot areas produced by C, M, J amounts of cyan, magenta, and yellow inks, then their values are given by the Demichel equations:

$$
\begin{aligned}
\bar{W} &= (1-C)\,(1-M)\,(1-J)\\
\bar{C} &= C\,(1-M)\,(1-J)\\
\bar{M} &= (1-C)\,M\,(1-J)\\
\bar{J} &= (1-C)\,(1-M)\,J\\
\bar{R} &= (1-C)\,M\,J\\
\bar{G} &= C\,(1-M)\,J\\
\bar{B} &= C\,M\,(1-J)\\
\bar{K} &= C\,M\,J
\end{aligned}
\tag{1}
$$

Weighing the measured XYZ tristimulus coordinates of these eight colors by their fractional dot area, one can predict the XYZ coordinates of the color produced by a CMJ ink combination.

$$
\begin{aligned}
X = {} & (1-C)(1-M)(1-J)X_w + C(1-M)(1-J)X_c + (1-C)M(1-J)X_m \\
& + (1-C)(1-M)J X_j + (1-C)MJ X_r + C(1-M)J X_g + CM(1-J)X_b \\
& + CMJ X_k
\end{aligned} \tag{2}
$$

Similar equations give the Y and Z components:

$$
\begin{aligned}
Y = {} & (1-C)(1-M)(1-J)Y_w + C(1-M)(1-J)Y_c + (1-C)M(1-J)Y_m \\
& + (1-C)(1-M)J Y_j + (1-C)MJ Y_r + C(1-M)J Y_y + CM(1-J)Y_b \\
& + CMJ Y_k \\
Z = {} & (1-C)(1-M)(1-J)Z_w + C(1-M)(1-J)Z_c + (1-C)M(1-J)Z_m \\
& + (1-C)(1-M)J Z_j + (1-C)MJ Z_r + C(1-M)J Z_y + CM(1-J)Z_b \\
& + CMJ Z_k
\end{aligned} \tag{3}
$$

These are known as the Neugebauer equations (Neugebauer 1937). It was found that for regular halftone screens which are rotated so that the cyan screen is 30 degrees from the magenta, and both are 15 degrees from the yellow, the dots are mixed almost as well as a random distribution. The Neugebauer equations can therefore be used to predict the CIE XYZ color coordinates of a color produced by C, M, J amounts of cyan, magenta, and yellow ink. Unfortunately, one usually wants to go the other way, to predict C, M, J from X, Y, Z.

Hardy & Wurzburg (1948), and Hardy & Dench (1948) describe analog computing circuitry which solves the Neugebauer equations for a color scanner.

A digital solution to the equations was given by Probboravsky and Pearson (1972), and also by Engeldrum (1975).

The main portion of the algorithm is as follows:

1. Assume a starting set of dot areas C, M, J.
2. Calculate the X, Y, Z tristimulus values using the Neugebauer equations.
3. Calculate the error from the desired values $\Delta X, \Delta Y, \Delta Z$.
4. If the error is within tolerance, then stop.
5. Calculate the nine partial derivatives of the Neugebauer equations to give the coefficients of three equations for the errors $\Delta X, \Delta Y, \Delta Z$ in terms of changes in the dot areas $\Delta C, \Delta M, \Delta J$.
6. Simultaneously, solve the three equations to get $\Delta C, \Delta M, \Delta J$.
7. Use $\Delta C, \Delta M, \Delta J$ to revise C, M, J and iterate from step 2.

Although the Neugebauer equations are typically written in terms of the CIE XYZ color basis, an alternate basis could be chosen such as monitor RGB coordinates. The equation for red would look like:

$$\begin{aligned} R = {} & (1-C)(1-M)(1-J)R_w + C(1-M)(1-J)R_c + (1-C)M(1-J)R_m \\ & + (1-C)(1-M)JR_j + (1-C)MJR_r + C(1-M)JR_g + CM(1-J)R_b \\ & + CMJR_k \end{aligned} \tag{4}$$

where R_w, R_c, R_m, R_j, R_r, R_g, R_b, and R_k are the measured amounts of the monitor red primary in the white, cyan, magenta, yellow, red, green, blue, and black produced by the printer. This can be found by measuring the XYZ coordinates of the printer's colors and converting them to the monitor RGB coordinates (typically a linear transformation).

The equations for the green and blue components are similar, with the *R*'s replaced by *G*'s or *B*'s.

Yule (1967) describes a solution to the Neugebauer equations, based on a number of simplifying assumptions including: the reflectance of an ink combination is equal to the product of the reflectance of its components; there is total reflectance of red light by the magenta and yellow inks, and total reflectance of green light by the yellow ink; and red light is completely absorbed by the cyan ink, green light is completely absorbed by the magenta, and blue is totally absorbed by the yellow.

With these aggressive assumptions, it is possible to determine the *C*, *M*, *J* dot areas required to produce an RGB color.

$$\begin{aligned} C &= 1-R \\ M &= 1-\frac{G}{G_c+(1-G_c)R} \\ J &= 1-\frac{B}{(B_c+(1-B_c)R)(B_m+(1-B_m)(1-M))} \end{aligned} \tag{5}$$

Undercolor Removal and the Neugebauer Equations

The discussion so far has considered the determination of the amounts of three inks: cyan, magenta, and yellow. When all three inks are present they absorb all colors yielding black. But often a fourth black ink is used. The black ink may be printed, in addition to the three-color process black, to get darker shadows, or it may be used to replace some of the three-color black to save ink and reduce gray balance sensitivity (Johnson 1985, Frost 1986, Holub 1989).

Undercolor removal and gray component replacement refer to the idea of removing some of the three-color process black and replacing it with the single black ink. Full undercolor removal is the case where all of the process black is replaced by black ink. Thus there is no region on the print where one can find all three of the cyan magenta and yellow inks together. There can only be two of these colors and the black ink.

The addition of the fourth ink would seem to complicate the problem further. There are 16 possible ink combinations, and the Neugebauer equations must be generalized to include these 16 terms. The terms now have a factor of either K or $(1-K)$ for when the black ink is or is not present respectively. The equations contain the products of four ink areas instead of three, and the equations are underdetermined, so there are many possible solutions. To complete the equations, one must decide how much of the black should arise from the fourth ink, and how much from the three-ink process black. There does not seem to be a consensus as to the best proportion of black ink.

Surprisingly, there is a case where the addition of a fourth black ink and a few simplifying assumptions can reduce the problem to the point where the Neugebauer equations can be solved analytically. This is the case where full undercolor removal is employed. Full undercolor removal and gray component replacement simplifies the problem by removing one of the inks. At any point on the print there are only two of the cyan, magenta and yellow inks along with some amount of black. This gives three cases but for each case half of the terms in the Neugebauer equations may be removed.

A further simplifying assumption is that the black ink is perfect, that all light is absorbed wherever the the black ink is present. This allows the removal of all terms containing black, once again cutting the number of terms in half. Under these assumptions each of the Neugebauer equation has only four terms.

While this simplified case admits the solution described below, full undercolor removal is not generally done. A problem is that the black ink is often not as dense as the three-color process black. Another problem is an abrupt transition at the neutral axis. In spite of these problems, the case is of interest because it allows a novel new and simple solution to the Neugebauer equations.

Using full undercolor removal, there are no more than three inks at any point on the page, two colors and black. If one assumes that the black ink is very good (so that when black is combined with other colors it is still just black), essentially removing all light, then there are only five possible ink colors at any point on the image:: the white paper, the first colored ink, the second colored ink, the combination of the two colors, and black.

Consider the case where the only inks present are cyan, magenta, and a perfect black. Then the equations reduce to the following:

$$
\begin{aligned}
X &= (1-C)(1-M)(1-K)X_w + C(1-M)(1-K)X_c \\
&\quad + (1-C)M(1-K)X_m + CM(1-K)X_b \\[6pt]
Y &= (1-C)(1-M)(1-K)Y_w + C(1-M)(1-K)Y_c \\
&\quad + (1-C)M(1-K)Y_m + CM(1-K)Y_b \\[6pt]
Z &= (1-C)(1-M)(1-K)Z_w + C(1-M)(1-K)Z_c \\
&\quad + (1-C)M(1-K)Z_m + CM(1-K)Z_b
\end{aligned}
\tag{6}
$$

Dividing out the $(1-K)$ factor yields the pair of equations:

$$
\begin{aligned}
& X\,((1-C)(1-M)Z_w + C(1-M)Z_c + (1-C)M\,Z_m + C\,M\,Z_b) \\
&\quad = Z\,((1-C)(1-M)X_w + C(1-M)X_c + (1-C)M\,X_m + C\,M\,X_b) \\[6pt]
& Y\,((1-C)(1-M)Z_w + C(1-M)Z_c + (1-C)M\,Z_m + C\,M\,Z_b) \\
&\quad = Z\,((1-C)(1-M)Y_w + C(1-M)Y_c + (1-C)M\,Y_m + C\,M\,Y_b)
\end{aligned}
\tag{7}
$$

Solving these for M gives the quadratic equation:

$$0 = a\,M^2 + \beta\,M + \gamma \tag{8}$$

where

$$
\begin{aligned}
a = {} & X\,((Y_c-Y_b)(Z_w-Z_m) - (Z_c-Z_b)(Y_w-Y_m)) \\
& + Y\,((Z_c-Z_b)(X_w-X_m) - (X_c-X_b)(Z_w-Z_m)) \\
& + Z\,((X_c-X_b)(Y_w-Y_m) - (Y_c-Y_b)(X_w-X_m))
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\beta = {} & X\,((Y_m-Y_w)(Z_b-Z_c)-(Z_m-Z_w)(Y_b-Y_c)-Z_wY_c+Z_cY_w-Z_bY_m-Z_mY_b) \\
& + Y\,((Z_m-Z_w)(X_b-X_c)-(X_m-X_w)(Z_b-Z_c)-X_wZ_c+X_cZ_w-X_bZ_m-X_mZ_b) \\
& + Z\,((X_m-X_w)(Y_b-Y_c)-(Y_m-Y_w)(X_b-X_c)-Y_wX_c+Y_cX_w-Y_bX_m-Y_mX_b)
\end{aligned}
$$

$$\gamma = X\,(Z_wY_c-Y_wZ_c)+Y(X_wZ_c-Z_wX_c)+Z(Y_wX_c-X_wY_c)$$

One can solve the quadratic to find M.

$$M = \frac{-\beta \pm \sqrt{(\beta^2-4a\gamma)}}{2a} \tag{10}$$

Then C is given by

$$C = \frac{Z(X_w + M(X_w - X_m)) - X(Z_w + M(Z_m - Z_w))}{X(Z_c - Z_w + M(Z_w - Z_c - Z_m + Z_b)) - Z(X_c - X_w + M(X_w - X_c - X_m + X_b))} \tag{11}$$

And finally K is given by

$$K = 1 - \frac{X}{(1-C)(1-M)X_w + C(1-M)X_c + (1-C)MX_m + CMX_b} \tag{12}$$

The other two cases of cyan-yellow-black and magenta-yellow-black can be solved in the same manner, and result in the same equations with the appropriate change in variables and subscripts.

One apparent problem with this scheme is that one must know which of the three cases one is dealing with in order to solve the correct set of equations. However the selection of the of the appropriate case depends on the relative strengths of the cyan, magenta and yellow, which is only known after solving the equations. This, however, is more of a theoretical than a practical problem. In practice one can solve the equations and then check for consistency. If necessary one can solve all three cases and the choose the appropriate one, although one can usually guess which case is most likely to be appropriate and can check it first to reduce computation.

Yule-Neilsen Correction

The Neugebauer equations are based on the assumption that the proportion of light modified by the inks is given by the dot area. This may be true for transparencies, but Yule and Nielsen (1951) have shown that for hardcopy prints the light passes through the ink dot pattern and into the paper. There it bounces around and becomes diffused before reemerging through the dot pattern. In this model the light passes through the dot pattern twice, and the result is to square the effectiveness. What is measured as the color of the ink is not the color of light passing once through the ink (as the Neugebauer equations assume), but rather the result of the light passing twice through the ink, the square of the color for light passing once. In order to use the Neugebauer equations, one should therefore use the square roots of the color coordinates. Yule points out that there are a number of reasons why this overly simplified model is not correct. Light may also enter the paper outside the dot and emerge from the dot interior, or enter through the dot and leave through the exposed paper, giving an "optical-dot-gain" effect. However, good empirical results are obtained if one modifies the Neugebauer equations by using color coordinates raised to the power $1/N$, where N is near 2, and depends on factors like the type of paper used. Viggiano (1985, 1990) suggests that better results can be obtained by applying the Yule-Neilsen method to narrow bands of the spectrum and integrating the results.

Instead of using $(X_i\ Y_i\ Z_i)$, where i indicates one of the basic colors ($i = w, c, m, j, r, g, b$, or k), use the modified coordinates (X_i', Y_i', Z_i') where

$$
\begin{aligned}
X_i' &= X_i^{1/Nx} \\
Y_i' &= Y_i^{1/Ny} \\
Z_i' &= Z_i^{1/Nz}
\end{aligned}
\tag{13}
$$

When solving for the $C\ M\ Y\ K$ amounts for color (X, Y, Z) use (X', Y', Z') in the Neugebauer equations:

$$
\begin{aligned}
X' &= X^{1/Nx} \\
Y' &= Y^{1/Ny} \\
Z' &= Z^{1/Nz}
\end{aligned}
\tag{14}
$$

This is essentially a coordinate transformation, where the new color coordinates are powers of the measured coordinates. After making the coordinate transformation, one then solves the Neugebauer equations using the new coordinates. Except for the primes on the coordinates, the equations are unchanged and can be solved by the method described above.

However, using a Yule-Neilsen correction also requires an alteration to the tone reproduction curve [TRC] adjustment of the printer. The TRC is a mapping between the color amount specification given to the printer and the density of the actual print. The printer technology may be highly non-linear, so some mapping is needed to tell what specification should be used to get a desired density. The TRC can be determined by printing halftone patterns for various specifications and measuring the densities of the results. Note that this includes both the effect of the printing technology (the mapping from specification to actual dot size) and the Yule-Neilsen effect (which makes the pattern look darker than the simple proportion of paper covered with ink). If compensation for the Yule-Neilsen effect is carried out in the modified Neugebauer equations, then this effect should be factored out of the TRC correction.

The Neugebauer equations, even with the Yule-Neilsen modification, are an approximation to a complex physical process and may not yield results of sufficient accuracy. One approach to improve the accuracy suggested by Laihanen (1988, 1989) is to use the results of the Neugebauer equations as a first-order approximation to the desired ink amounts. To this approximation one adds a correction term which is derived from interpolating a small table of measured differences between the predicted and actual color.

Conclusions

The Neugebauer equations provide a model for color printing of ink on paper using rotated halftone screens. Solving the equations predicts the ink dot areas needed to generate a desired color. The equations can be solved by numerical methods, but for the special case of full undercolor removal and a perfect black ink the equations simplify to quadratic forms which can be solved analytically. The Yule-Neilsen correction may yield better modeling of the printer behavior. This correction can be treated as a change of coordinates and does not interfere with the method of solving the equations.

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